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Difference of Observability Between Classical Electromagnetic and Gravitational Gauge Fields[1,2]

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An analysis of the observability of the classical electromagnetic gauge field based in its quantum effects shows that this is physically determined up to equivalences. By contrast a similar analysis of the gravitational gauge field from Einstein's General Relativity theory shows that this field is univocally determined by the trajectories of material particles provided they feel only that gravitational field, and its proper gravitational and quantum effects are negligible. This difference of observability in both kinds of gauge fields is caused by the attachment of the gravitational field in the Einstein theory to the space-time, and this difference must be taken into account to formulate unified gauge theories with both kinds of fields.

1. INTRODUCTION

Calling *gauge field* any connection defined in a principal fiber bundle on a space-time manifold, the Leví–Civita connection Γ_0 of any space-time $(\mathscr{M}, g)$ is obviously a gauge field, and therefore, since such a connection describes the gravitational field corresponding to $(\mathscr{M}, g)$ in Einstein's General Relativity theory, the latter is also a gauge field. On the other hand, classical electromagnetic fields in quantum mechanics are described by connections defined in principal fiber bundles on open and connected submanifolds of the manifold of a space-time (Greub and Petry, 1975). Accordingly,

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in this picture the electromagnetic field is also a gauge field. But, up to which point are these gauge fields, associated with the physical gravitational or electromagnetic fields, observable in both theories? Or, can there exist experimentally indistinguishable gravitational or electromagnetical gauge fields? The answer to these two questions is the aim of the present paper. In Section 2 we prove the existence of an infinity of electromagnetic gauge fields defined in any principal fiber bundle corresponding to the same values of the electric and magnetic fields, and we give a characterization of their equivalence classes. In Section 3 we see that the results of the Aharonov–Bohm experiment (1959) imply the observability of the electromagnetic field equivalence classes and the experimental indistinguishability of the gauge fields belonging to the same class. In Section 4 we prove that the knowledge of the trajectories of the material particles whose masses and quantum effects are negligible under the effect only of a gauge gravitational field determine the latter in a unique way. Finally, in Section 5 we analyze the causes and implications of the different degree of observability of the gravitational and electromagnetical gauge fields disclosed in Sections 3 and 4.

2. GAUGE FIELDS ASSOCIATED WITH CLASSICAL ELECTROMAGNETIC FIELDS COMPATIBLE WITH QUANTUM MECHANICS

The classical electromagnetic field is described (Sachs and Wu, 1977) by means of a real, closed 2-form F ($dF = 0$) defined in an open connected submanifold M of the manifold $\mathcal{M}$ of a space-time $(\mathcal{M}, g)$ in such a way that $\mathcal{M} - M$ is the set of points in $\mathcal{M}$ in which either the electrical sources are singular or there are magnetic sources. The restriction of g to M induces a space-time structure in M.

The behavior of a punctual classical particle under a certain field F can be described (Sternberg, 1978) by a mechanical system whose symplectic form in the phase space $T^*(M)$ is

$$\omega = d\theta - (e/c)\Pi_{T^*}^{*}F$$

where θ is the canonical 1-form of $T^*(M)$ and Π_{T^*} is the canonical projection of $T^*(M)$ on M.

Consequently, as Sniatycki (1974) has shown, the condition of prequantization (Kostant, 1970) for such a mechanical system implies that

$$\epsilon^2[(e/hc)F] \in i^2(H^2(M, \mathbb{Z})) \tag{2.1}$$

i^2 being the homomorphism of the Čech cohomology group $\check{H}^2(M, \mathbb{Z})$ in the $\check{H}^2(M, \mathbb{R})$ induced by the homomorphism of inclusion of $\mathbb{Z}$ in $\mathbb{R}$, $[(e/hc)F]$ being the Rham cohomology class to which the closed 2-form $(e/hc)F$ belongs, and ϵ^2 being the isomorphism of $H^2_{dR}(M, \mathbb{R})$ in $\check{H}^2(M, \mathbb{R})$ established by Rham's theorem. Now, property (2.1) is the necessary and sufficient condition for the existence of a connection defined in a principal fiber bundle $P(M, U(1))$ with curvature $(e/hc)\Pi_P^* F\chi$, χ being the element of the Lie algebra tangent to the curve $w: \mathbb{R} \to U(1)$ with $w(\alpha) = e^{2\pi i\alpha}$ for every $\alpha \in \mathbb{R}$ (Asorey, 1978); summarizing, *the prequantization condition for the mechanical system associated to a particle subject in M to a field F implies the existence of a gauge field in a principal fiber bundle P(M, U(1)) whose curvature is* $(e/hc)\Pi_P^* F\chi$. In this sense, only those electromagnetic fields which satisfy property (2.1) are compatible with quantum mechanics.

But generally, the gauge field associated in that way to the field F satisfying (2.1) is not unique, for ω being the 1-form of connection of one such Γ defined in a fiber bundle $P(M, U(1))$, gauge fields Γ' whose 1-form of connection ω' satisfy

$$\omega' = \omega + \Pi_P^*\left(\frac{d\xi}{2\pi i\xi}\right)\chi \tag{2.2}$$

where ξ is a function of M on $\mathbb{C}$ with values in $U(1)$, give rise to the same field F. Since, in general, an infinity of functions ξ of M on $\mathbb{C}$ with values in $U(1)$ and $d\xi \neq 0$ exist, *there exists, too, an infinity of gauge fields with the same electric and magnetic fields*. However, given that all gauge fields Γ' defined by (2.2) are equivalent [understanding the equivalence of two gauge fields defined on M with the same gauge group G as the existence of an M isomorphism (ρ, id_G) between the principal fiber bundles, in which the two gauge fields are defined, which transforms one into the other], there is room for asking whether at least all gauge fields giving rise to the same F are equivalent. In general this is not true either, as we shall see in the sequel.

Since $U(1)$ is an Abelian group, the element $k_\gamma{}^\Gamma$ of the holonomy group of a gauge field Γ defined on a principal fiber bundle $P(M, U(1))$ in a point $u \in P$ corresponding to a closed curve γ with $\gamma(0) = \gamma(1) = \Pi_P(u)$ is independent of the element u of $\Pi_P^{-1}(\Pi_P(u))$ considered. Indeed, γ_u being the horizontal lift with respect to Γ of γ in P with $\gamma_u(0) = u$, for any $\bar{u} \in P$ with $\Pi_P(u) = \Pi_P(\bar{u})$ the horizontal lift $\gamma_{\bar{u}}$ of γ with respect to Γ in P with $\gamma_{\bar{u}}(0) = \bar{u}$ satisfies

$$\gamma_{\bar{u}}(t) = \gamma_u(t)\cdot a$$

a being the element of $U(1)$ with $\bar{u} = ua$. In this way

$$\gamma_{\bar{u}}(1) = \gamma_u(1)\cdot a = \gamma_u(0)\cdot k_\gamma{}^\Gamma \cdot a = \gamma_{\bar{u}}(0)\cdot k_\gamma{}^\Gamma$$

i.e., $k_\gamma{}^\Gamma$ equals the element of the holonomy group of Γ in $\bar{u}$ corresponding to γ.

Proposition 1. Two gauge fields Γ and Γ' defined on M with gauge group $U(1)$ are equivalent iff for any closed curve γ of M the equality

$$k_\gamma{}^\Gamma = k_\gamma{}^{\Gamma'} \tag{2.3}$$

holds.

Proof. If Γ and Γ' are equivalent, an M isomorphism $(\rho, id_{U(1)})$ exists between the principal fiber bundles $P(M, U(1))$ and $P'(M, U(1))$, in which Γ and Γ' are defined, which maps Γ on Γ'. Accordingly, for any closed curve γ of M holds that $\boldsymbol{\gamma}$ being any horizontal lift of γ with respect to Γ in P, $\rho(\boldsymbol{\gamma})$ is a horizontal lift of γ with respect to Γ' in P'. Then, since

$$\rho(\boldsymbol{\gamma})(1) = \rho(\boldsymbol{\gamma}(1)) = \rho(\boldsymbol{\gamma}(0)\cdot k_\gamma{}^\Gamma) = \rho(\boldsymbol{\gamma}(0))\cdot k_\gamma{}^\Gamma = \rho(\boldsymbol{\gamma})(0)\cdot k_\gamma{}^\Gamma$$

(2.3) holds.

Conversely, let us assume that Γ and Γ' satisfy (2.3) for any closed curve γ in M. Choosing $u_0 \in P$ and $u_0' \in P'$ with $\Pi_P(u_0) = \Pi_{P'}(u_0')$ let us define the mapping ρ of P in P' in the following way. If c is any curve of M with $c(0) = \Pi_P(u_0)$ and c_{u_0}, $c_{u_0'}$ are its horizontal lifts with respect to Γ in P and with respect to Γ' in P' such that $c_{u_0}(0) = u_0$ and $c_{u_0'}(0) = u_0'$, let us make

$$\rho(c_{u_0}(t)) = c_{u_0'}(t)$$

for all $t \in [0, 1]$. Since M is connected arcwise, there exists, for each $x \in M$, a curve c with $c(0) = \Pi_P(u_0)$ and $c(1) = x$, and therefore in each fiber of P there is some element for which ρ is defined. Let us call P_0 the set of elements of P in which ρ is defined in such way. Defining ρ for the rest of the elements of P in such way that

$$\rho(u\cdot a) = \rho(u)\cdot a$$

for any $u \in P_0$ and $a \in U(1)$, it is trivial to see that $(\rho, id_{U(1)})$ is an M homomorphism of P in P'. And obviously ρ transforms Γ in Γ'. In the same way we can construct an M homomorphism $(\rho', id_{U(1)})$ of P' in P which transforms Γ' into Γ. But from the construction of ρ and ρ' it follows that $\rho_0\rho' = id_{P'}$ and $\rho'_0\rho = id_P$, which implies that $(\rho, id_{U(1)})$ is an M isomorphism of $P(M, U(1))$ in $P'(M, U(1))$ and therefore the equivalence of Γ and Γ'. ∎

Calling ΩX_{x_0} the H group of closed curves in M with beginning and end in a point x_0 of M, it follows from the previous proposition that the mapping κ of the set of equivalence classes of gauge fields on M with gauge group $U(1)$ onto H-Hom $(\Omega M_{x_0}, U(1))$, defined for the class of any gauge field Γ by

$$\kappa([\Gamma])(\gamma) = k_\gamma{}^\Gamma$$

for all curves $\gamma \in \Omega M_{x_0}$, is injective.

Now then, Kostant (1970) has proved that the image through κ of the equivalence classes of the gauge fields which give rise to the same 2-form F of M (in this case to the same electric and magnetic fields) is an orbit of the action of Hom $(\pi_1(M), U(1))$ on H-Hom $(\Omega M_{x_0}, U(1))$ defined by

$$l \cdot \Pi(\gamma) = l(\gamma) \cdot \Pi([\gamma])$$

for any $l \in H\text{-Hom}\,(\Omega M_{x_0}, U(1))$, $\Pi \in \text{Hom}\,(\pi_1(M), U(1))$ and $\gamma \in \Omega M_{x_0}$. Since this action is free, *there are Card* {Hom $(\pi_1(M), U(1))$} *classes of gauge fields which give rise to the same electric and magnetic fields.*

If M is simply connected, $\pi_1(M) = 0$ and Hom $(0, U(1)) = \{1\}$. Therefore, in this case there is only one class of gauge fields giving rise to one F. But as, in general, Hom $(\pi_1(M), U(1)) \neq \{1\}$, there can be more, and consequently, not all gauge fields with the same electric and magnetic fields are equivalent.

3. OBSERVABILITY OF THE CLASSICAL ELECTROMAGNETIC GAUGE FIELDS

Let us consider

$$M = \mathbb{R}^2 \times A$$

A being the complementary in $\mathbb{R}^2$ of the dashed region in Figure 1. Since $\pi_1(M) = \mathbb{Z}$, if γ_0 is any simple closed curve of M whose graph is the immersion in M of the subset c_0 of A pointed out in Figure 1 and $x_0 = \gamma_0(0) = \gamma_0(1)$, each element $h \in \text{Hom}\,(\pi_1(M), U(1))$ is perfectly labeled by $h([\gamma_0]) \in U(1)$.

Proposition 2. For each $a \in U(1)$ there exist a unique class of gauge fields on M with gauge group $U(1)$ whose associated field F is null and such that

$$\kappa(\Upsilon_a)(\gamma_0) = a \tag{3.1}$$

This class Υ_a uniquely associated to each $a \in U(1)$ is such that also any curve $\gamma \in \Omega M_{x_0}$ homotopic to γ_0 satisfies

$$\kappa(\Upsilon_a)(\gamma) = a$$

Proof. Obviously the canonical connection $\mathring{\Gamma}$ of $M \times U(1)$ has null curvature and therefore gives rise to a null F. Now, since $\kappa([\mathring{\Gamma}])$ is the neutral element of H-Hom $(\Omega M_{x_0}, U(1))$, if h is the only element of Hom $(\pi_1(M), U(1))$ with $h([\gamma_0]) = a$, $\kappa([\mathring{\Gamma}]) \cdot h \in H\text{-Hom}\,(\Omega M_{x_0}, U(1))$ satisfies

$$\{\kappa([\mathring{\Gamma}]) \cdot h\}(\gamma_0) = \kappa([\mathring{\Gamma}])(\gamma_0) \cdot h([\gamma_0]) = a$$

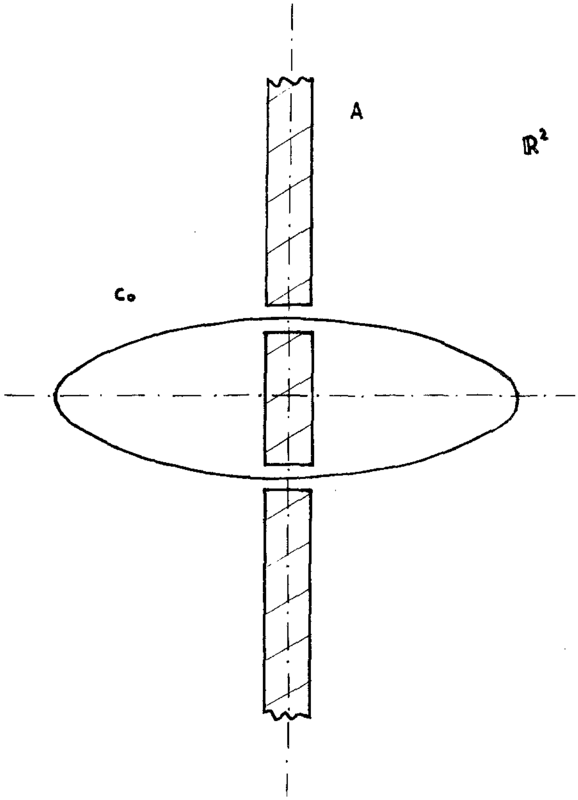

Fig. 1.

Υ_a being the class of gauge fields of M with gauge group $U(1)$ such that $\kappa(\Upsilon_a) = \kappa([\mathring{\Gamma}])\cdot h$, Υ_a satisfies (3.1) and is obviously unique. Lastly, γ being any curve of ΩM_{x_0} homotopic to γ_0, it follows that

$$\{\kappa([\mathring{\Gamma}])\cdot h\}(\gamma) = \kappa[(\mathring{\Gamma}])(\gamma)\cdot h([\gamma]) = a \qquad \blacksquare$$

The element $a \in U(1)$ is called by Wu and Yang (1975) "*phase factor*" of any Γ of Υ_a.

Since in classical electromagnetism the physically relevant quantity is the field F, all classes $\{\Upsilon_a; a \in U(1)\}$ of gauge field with $F = 0$ give rise, in the classical picture, to identical observable effects. But this is different in quantum mechanics, because provided the dimensions of the dashed part of Figure 1 are suitable, the observed results in the Aharonov–Bohm experiment (Chambers, 1960) are different for each phase factor, and those corresponding to every phase factor are observed. This implies that *each class of gauge fields* Υ_a *is observable*, since its effects are experimentally observed, and are distinguishable from those caused by the other classes Υ_b with $b \neq a$.

If F is not null, the phase factor is no longer independent of the simple loop chosen, but the result holds.

This result is not peculiar to the manifold $M = \mathbb{R}^2 \times A$, for the same result would follow with any manifold of the form $M = \mathbb{R} \times B$, B being an open connected submanifold of $\mathbb{R}^3$ with $\pi_1(B) = \mathbb{Z}$. Analogous results follow when $\pi_1(B) = \mathbb{Z}^n$, though in this case there are n phase factors.

Consequently, extrapolating these results it can be stated that for any open and connected submanifold M of any space-time $(\mathscr{M}, g)$ *the classes of electromagnetic gauge fields defined on M are physically distinguishable*, since when they give rise to different fields F they are so by the laws of classical electromagnetism, and when they give rise to the same F they are made so by the result of an experiment similar to that of Aharonov–Bohm.

On the contrary, the Aharonov–Bohm experiment states that all equivalent gauge fields yield the same observable results, for the latter depend only on the phase factor $a \in U(1)$ which is the same for all gauge fields Γ of the class Υ_a. In principle it could be thought that this degeneracy might be removed by a different physical experiment, but this is not so due to the gauge invariance of the wave equations and physical observables of quantum particles subject to the action of a classical electromagnetic field. Indeed, Γ being an electromagnetic gauge field defined in a principal fiber bundle $P(M, U(1))$ whose connection 1-form is ω, and ϕ being a section of a complex vector fiber bundle $E(M, P, \mathbb{C}^n)$ associated to P which is a solution of the wave equation of a given quantum particle under the action of Γ, for any function ξ of M in $\mathbb{C}$ with values in $U(1)$ it holds that $\phi' = \xi\phi$ is a solution to the same wave equation for the gauge field Γ' equivalent to Γ whose connection form ω' is defined by (2.2). And, since for any vector field X of M

$$\nabla'_X\phi' = \xi \cdot \nabla_X\phi$$

∇_X and ∇'_X being the covariant derivatives with respect to Γ and Γ' in the direction of X, for any pseudo-Hermitian metric H defined in E it holds that

$$\begin{aligned} H(\phi, \phi) &= H(\phi', \phi') \\ H(\nabla_X\phi, \phi) &= H(\nabla'_X\phi', \phi') \\ H(\nabla_X\phi, \nabla_X\phi) &= H(\nabla'_X\phi', \nabla'_X\phi') \end{aligned}$$

which implies that the states (ϕ, Γ) and (ϕ', Γ') of the particle and electromagnetic field are physically indistinguishable. Now, since all gauge fields of P equivalent to Γ are obtained in this way, and since a similar reasoning can be used for the pairs of equivalent electromagnetic gauge fields when they are defined in different principal fiber bundles, *equivalent electromagnetic gauge fields are physically indistinguishable*, in the same way as in quantum mechanics all vectors in the same ray of Hilbert space are so. And, following the same analogy, in the same way in which in quantum mechanics a pure state of a quantum system is considered described by a ray in a Hilbert space, and not by any particular one of its vectors, also in electromagnetism it must be considered that *a state of the classical electromagnetic field is described by an equivalence class of gauge fields, and not by any particular gauge field of this class.*

4. OBSERVABILITY OF THE GRAVITATIONAL GAUGE FIELD IN GENERAL RELATIVITY

A space-time $(\mathcal{M}, g)$ is a four-dimensional connected and orientable manifold $\mathcal{M}$ in which there is defined a (Lorentzian) metric g with signature $(-+++)$ such that $\mathcal{M}$ is g time orientable. The unique linear connection on M torsionless and metric with respect to g is the Levi–Civita connection Γ_0.

In Einstein's General Relativity theory the gravitational field is described by the Levi–Civita connection Γ_0 in a space-time $(\mathcal{M}, g)$ whose geodesics $\gamma: [0, 1] \to \mathcal{M}$ such that for any $t \in [0, 1]$, $g(\dot{\gamma}_t, \dot{\gamma}_t) < 0$ with $\dot{\gamma}_t = \gamma_{*t}(1)$, verify that $\gamma([0, 1])$ is a segment of a trajectory of a possible material test particle (submitted only to the action of that gravitational field and neglecting its own gravitational field and quantum effects).

Therefore in that theory if two gravitational fields Γ_0 and $\bar{\Gamma}_0$ associated to two space-times $(\mathcal{M}, g)$ and $(\mathcal{M}, \bar{g})$ defined on the same manifold $\mathcal{M}$ are physically indistinguishable, they give rise to the same trajectories for all material test particles. And hence the following proposition holds.

Proposition 3. The gravitational fields Γ_0 and $\bar{\Gamma}_0$ associated to two space-times $(\mathcal{M}, g)$ and $(\mathcal{M}, \bar{g})$ are physically indistinguishable iff there is a real positive constant λ such that $\bar{g} = \lambda g$; and in this case $\bar{\Gamma}_0 = \Gamma_0$.

Proof. Let p be any $\mathcal{M}$ point and v a vector of $T_p(\mathcal{M})$, the tangent space to $\mathcal{M}$ at p. Let τ_v be the endomorphism of $T_p(\mathcal{M})$ defined by

$$\tau_v(w) = (\bar{\nabla}_v - \nabla_v)X$$

for any vector field X of $\mathcal{M}$ with $X_p = w$, where $\bar{\nabla}_v$ and ∇_v are the covariant derivatives in the direction v with respect to $\bar{\Gamma}_0$ and Γ_0. For any $v \in T_p(\mathcal{M})$ with $g(v, v) < 0$ there exists a curve γ geodesic with respect to g with $\gamma(1/2) = p$, and $\gamma(t) \neq \gamma(t')$ for $t \neq t'$, $\dot{\gamma}_{1/2} = v$, and $g(\dot{\gamma}_t, \dot{\gamma}_t) < 0$ for all $t \in [0, 1]$. Therefore, if Γ_0 and $\bar{\Gamma}_0$ are indistinguishables, due to the fact that $\gamma([0, 1])$

is a segment of a trajectory of a possible test particle moving under Γ_0, it must also be so under the action of $\bar{\Gamma}_0$, and therefore there will exist also a geodesic $\bar{\gamma}$ of $(\mathscr{M}, \bar{g})$ with $\bar{\gamma}([0, 1]) \subset \gamma([0, 1])$, $\dot{\bar{\gamma}}(1/2) = v$, $\bar{\gamma}(1/2) = p$, and $\bar{g}(\dot{\bar{\gamma}}_t, \dot{\bar{\gamma}}_t) < 0$ for all $t \in [0, 1]$. Now $\mathscr{M}$ is Haussdorff, hence $\gamma([0, 1])$ and $\bar{\gamma}([0, 1])$ are closed sets of $\mathscr{M}$ and then there are two vector fields X and $\bar{X}$ of $\mathscr{M}$ with $X_{\gamma(t)} = \dot{\gamma}_t$ and $\bar{X}_{\bar{\gamma}(t)} = \dot{\bar{\gamma}}_t$ for all $t \in [0, 1]$. Hence, there is a function f: $[0, 1] \to \mathbb{R} - \{0\}$ verifying $\bar{X}_{\bar{\gamma}(t)} = f(t) X_{\bar{\gamma}(t)}$ for every $t \in [0, 1]$. And because

$$df/dt(1/2)v + \bar{\nabla}_v X = \bar{\nabla}_v(\bar{X}) = 0 \text{ and } \nabla_v X = 0$$

$$\tau_v(v) = \bar{\nabla}_v X - \nabla_v X = -df/dt(1/2)v = \alpha_v v \tag{4.1}$$

where $\alpha_v = -df/dt(1/2)$.

Now if $v \in T_p(\mathscr{M})$ verifies $g(v, v) < 0$, it verifies also $\bar{g}(v, v) < 0$, and the sets $V_p = \{v \in T_p(\mathscr{M}); g(v, v) < 0\}$ and $\bar{V}_p = \{v \in T_p(\mathscr{M}); \bar{g}(v, v) < 0\}$ are the same. Therefore, if we consider the unique topology compatible with any norm in $T_p(\mathscr{M})$, the boundary of V_p, which is the set of $v \in T_p(\mathscr{M})$ with $\bar{g}(v, v) = 0$], is the same as that of V_p [set of $v \in T_p(\mathscr{M})$ with $\bar{g}(v, v) = 0$]. Because of this for all $v \in T_p(\mathscr{M})$ with $g(v, v) = 0$ it happens also that $\bar{g}(v, v) = 0$ and vice versa, i.e., the light cones of g_p and $\bar{g}_p$ are the same. But in this case we have $\bar{g}_p = \lambda_p g_p$ with a $\lambda_p > 0$, and therefore there is a function λ: $\mathscr{M} \to (0, \infty)$ with $\lambda(p) = \lambda_p$, $p \in \mathscr{M}$. For any vectors $v, v', w \in T_p(\mathscr{M})$ we have, therefore,

$$2g(\tau_v v', w) = g(v, w)v'(\log \lambda) + g(v', w)v(\log \lambda) - g(v, v')w(\log \lambda) \tag{4.2}$$

If v and v' are two vectors of $T_p(\mathscr{M})$ with $g(v, v) = g(v', v') = 0$ and $g(v, v') <$ 0, as $g(v + v', v + v') < 0$, from (4.1) and (4.2) we obtain for all $\eta \in \mathbb{R}$

$$2g[\tau_{v+\eta v'}(\eta v' + v), v] = 2\eta\alpha_{v+\eta v'} g(v, v') = 2\eta^2 g(v, v')v'(\log \lambda)$$

$$2g[\tau_{v+\eta v'}(v + \eta v'), v'] = 2\alpha_{v+\eta v'} g(v', v) = 2v(\log \lambda)g(v, v')$$

which implies

$$\eta v'(\log \lambda) = v(\log \lambda) = \alpha_{v+\eta v'}$$

But as $v(\log \lambda)$ is η independent, all sides of the above equality must be zero, and because the set of vectors $v \in T_p(\mathscr{M})$ with $g(v, v) = 0$ is a system of generators of $T_p(\mathscr{M})$, this implies $d(\log \lambda) = 0 = d\lambda$. Hence, because $\tau_v w = 0$ for any $v, w \in T_p(\mathscr{M})$, we have $\bar{\nabla}_v = \nabla_v$; i.e., $\bar{\Gamma}_0 = \Gamma_0$. On the other hand, the connectedness of $\mathscr{M}$ implies that λ is a constant in $\mathscr{M}$. Lastly, if $\bar{\Gamma}_0 = \Gamma_0$, it is obvious that the gravitational fields described by both gauge fields are physically equivalent. ∎

As a consequence of the above proposition the gauge field of Einstein's General Relativity theory is totally observable: to know it completely it is enough, for example, to study the trajectories of the test particles under the action of this field, because there is only one gravitational gauge field which gives rise to them.

5. CONCLUSIONS AND DISCUSSION

From Sections 3 and 4 it follows the existence of a structural difference between the observability of the classical electromagnetic gauge field and that of gravitational gauge field of General Relativity, because this is physically directly observable, while on the other one we can observe only its equivalence classes, but the gauge field itself is not directly observable.

This structural difference is due to the attachment of the General Relativity gravitational field to the space-time $(\mathcal{M}, g)$, specifically to the tangent bundle of $\mathcal{M}$, which makes it a very singular gauge field, because the other possible physical gauge fields existing in nature are not so connected to the space-time, and hence for them it is to be expected the same as for the electromagnetic field, i.e., that its equivalence classes be observable, since the different gauge fields in each class are not, by the same reason why they are not observable either for the electromagnetic field as we saw at the end of Section 3. However, the practical realization of Aharonov–Bohm-type experiments for the other gauge fields to check the observability of its equivalence classes meets technical difficulties (Wu and Yang, 1975).

As a consequence of all this one can conclude that in spite of the common gauge character of all fundamental interactions, gravitation exhibits peculiar characteristics, coming from its special attachment to space-time. Therefore, the unified field theories including the gravitational field must explain this difference of the observability of the different unified gauge fields if they are to describe correctly the individual effects of each one in absence of the others; and this confers to the gravitational field a peculiar character in these theories, which is also a consequence of the universal character of this field.

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